In discrepancy between the traditional Fuzzy logic and inductive

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A R T I C L E I N F O

A B S T R A C T

In three case studies on notions of fuzzy logic and fuzzy set theory (Dubois–Prade's gradual elements, the entropy of a fuzzy set, and aggregation operators), the pa- per exemplifies methodological differences between traditional and deductive fuzzy logic. While traditional fuzzy logic admits various interpretations of membership degrees, deductive fuzzy logic always interprets them as degrees of truth preserved under inference. The latter fact imposes several constraints on systems of deductive fuzzy logic, which need not be followed by mainstream fuzzy logic. That makes deductive fuzzy logic a specific area of research that can be characterized both methodologically (by constraints on meaningful definitions) and formally (as a specific class of logical systems). An analysis of the relationship between deductive and traditional fuzzy logic is offered.

1. Introduction

In this paper we shall focus on a sub-area of FLn that studies or uses *formal deductive systems of* fuzzy logic. Prototypical examples of such systems are those centered around H´ajek's basic fuzzy logic BL of continuous t-norms, including for instance LÃ ukasiewicz, G¨odel, and product logics, the logics MTL, LÃ Π, etc., both propositional (Atanassov, 2005) and first- or higher-order . The area also covers those parts of fuzzy mathematics (i.e., of FLb) which are built as deductive axiomatic theories based on these formal fuzzy logics. To avoid a conflict of terms, we shall call this area *deductive fuzzy logic* (FLd). Other parts of FLn and FLb will in the present paper be labeled *traditional fuzzy logic* (FLt), as the latter has a much longer tradition than the relatively newer FLd.

The aim of this paper is to point out and analyze certain fundamental differences between FLd and FLt.   The differences are illustrated in three case studies, regarding respectively:

1. Dubois and Prade's notion of fuzzy element
2. The notion of entropy of fuzzy sets
3. Aggregation of fuzzy data

Since the paper is methodological rather than technical, I omit most technical details and focus on the   analysis   of   the   principles   behind   the approaches of FLt and FLd.   I assume that the reader has a basic knowledge of some formal system of fuzzy logic, for instance H´ajek's logic BL of continuous t-norms. Here I only briefly recapitulate some characteristic features of deductive fuzzy logics, which will be of importance for further considerations:

• Deductive fuzzy logic is a kind of (many-valued) *logic.* Therefore, like other kinds of logics, it primarily studies preservation of some quality ("truth") of propositions under inference. In the particular case of formal *fuzzy logic*, the quality is *partial truth,* i.e., the *degrees* of truth.   Thus, deductive fuzzy logic interprets membership degrees exclusively as degrees of truth of the membership predication.   In this it differs from the rest of traditional fuzzy logic, which admits various interpretations of membership degrees.

• As a kind of *formal or symbolic logic*, FLd strictly distinguishes syntax from semantics.   In syntax, deductive fuzzy logic works with some fixed language composed of propositional connectives, quantifiers, predicate and function symbols, and vari- ables. The symbols (and formulae built up from these symbols) are then interpreted in semantically models, which are composed of usual fuzzy sets and fuzzy relations of FLt. In this way the formulae of the symbolic language formally describe actual fuzzy sets.

• FLd is based on the axiomatic method and works in the formal deductive way. Valid statements about fuzzy sets are derived in an axiomatic theory,

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through iterated application of sound rules of a particular system of deductive fuzzy logic. Since FLd employs non-classical many-valued logic, formal theories in FLd can have some peculiar features, which are not met in standard axiomatic theories of FLt or classical mathematics.

• Most systems of FLd impose specific constraints on some of its components. For example, most systems of formal fuzzy logic require that conjunction be realized as a left-continuous t-norm, and are much less interested in other conjunctive operators studied in FLt. A partial explanation of this selectiveness of FLd will be elaborated in the present paper. It can be shown that such restrictions largely follow from the features of FLd listed above (viz the interpretation of degrees in terms of truth, the study of partial truth preservation, formal deducibility, etc.).

A further explanation and illustration of these points, as well as an analysis of the difference between FLd and FLt which results from the above features of FLd, will be given in the following sections. I will argue that FLd is a rather sharply delimited area of FLt, and that the agenda of FLd differs significantly from that of FLt. Therefore, to avoid confusion in fuzzy set theory, we should clearly distinguish between their respective areas of competence.

Case study 1: Dubois and Prade's gradual elements

Dubois and Prade have introduced the notions of gradual element and gradual set by the following definitions:

Definition 1. Let S be a set and L a complete lattice with top 1 and bottom 0. A *fuzzy* (or *gradual) element e* in S is identified with a (partial) assignment function $ae: L \setminus \{0\} \to S$.

Definition 2. A *gradual subset G* in S is identified with its assignment function $aG : L \setminus \{0\} \to 2S$. If S is fixed, we may simply speak of *gradual sets.*

The assignment function of a gradual element need not be monotone or injective (cf. the middle points of certain asymmetric fuzzy intervals). Fuzzy elements of this kind are met in many real-life situations (e.g., the average salary of older people). Gradual elements and gradual sets are claimed by the authors to be a missing primitive concept in fuzzy set theory.

The authors proceed to define the fuzzy set induced by a gradual set, the membership of a gradual element in a fuzzy set, etc. As these notions are not important for our present case, I refer the reader to the original article. We shall only notice that the operations proposed for gradual sets are defined cut-wise (with possible rearrangements of cuts in the case of complementation).

The declared motivation for introducing gradual elements is to distinguish *impreciseness* (i.e., intervals) from *fuzziness* (i.e., gradual change

from 0 to 1). As implicit in, a general guideline for definitions of fuzzy notions should be the following principle (we shall call it the *principle of cuts)*:

Principle of cuts: *The α-cuts of a fuzzy notion FX should be instances of the corresponding crisp notion X,* i.e., the fuzzy version *FX* of a crisp notion *X* should be defined in such a way that the α-cuts of *FX*'s are *X*'s. Thus the fuzzy counterpart of the notion of element is exactly the fuzzy element of Definition 1, that of the notion of set is the gradual set of Definition 2, etc.

The definitions of gradual sets and gradual elements are clearly sound and the notions will probably prove to be of considerable importance for FLt. Let us see if they can be represented in FLd as well. A more detailed analysis of this question has been done in; here we extract its important parts:

Apparently there are no direct counterparts of gradual elements or sets among the primitive concepts of current propositional or first-order fuzzy logics. Nevertheless, it can be shown that gradual elements and gradual sets are representable in *higher-order fuzzy logic* or *simple fuzzy type theory.* For technical details of the representation see; here we only sketch the construction:

1. By the comprehension axioms of higher-order fuzzy logic, the notions of crisp kernel, fuzzy subset, fuzzy powerset, and crisp function are definable in higher-order fuzzy logic (freely available primer for details).

2. By a standard construction, an internalization of truth degrees is definable in higher-order fuzzy logic (for the details of the construction and some meta-mathematical provisos). The lattice that represents truth degrees within the theory is defined as L = Ker(Pow(*{a}*)), i.e., the kernel of the power set of the crisp singleton of any element *a* of the universe of discourse. (In fuzzy type theory of, this step can be omitted, since the set of truth values is a primitive concept there.)

3. Since Definitions 1 and 2 need no further ingredients beyond those listed in items (1)–(2), crisp functions from L to the domain of discourse or its powerset represent respectively the notions of gradual element and gradual set in higher-order fuzzy logic. By similar means, all other notions defined in can be defined in higher- order fuzzy logic as well (Esteva, 2001).

In particular, the definitions of gradual elements and gradual sets in the standard framework of higher-order logic (or *fuzzy class theory)* run as follows:

Definition 3. A fuzzy element of *S (*in higher-order fuzzy logic) is any (second-order) class *E* such that Crisp *E* & Δ(Dom *E* ⊆ L \ *{∅}*) & Δ(Rng *E* ⊆ *S*) & Fnc *E .*

Definition 4. A gradual subset of *S in* higher-order fuzzy logic is any (second-order) class *G* such that Crisp *G* & Δ(Dom *E* ⊆ L \ *{∅}*) & Δ(Rng *E* ⊆ Ker Pow *S*) & Fnc *G.*

In this way, the FLt notions of gradual element and gradual set can also be defined in FLd of higher order. However, their rendering in FLd is not very satisfactory. First, the formal representatives in FLd of the simple FLt notions are rather complex—namely certain very special second-order predicates, whose relationship to traditional fuzzy sets (i.e., first-order predicates) is far from perspicuous. Although this presents no obstacle to handling them in the formal framework of higher-order fuzzy logic, the apparatus of FLd does not much simplify working with these notions (unlike it does with traditional fuzzy sets), since they are represented by crisp functions like in their semantic treatment by FLt. Considering the fundamental role fuzzy elements are to play in Dubois and Prade's recasting of fuzzy set theory, it would certainly be desirable to have fuzzy elements and gradual sets rendered more directly in FLd—as primitive notions rather than defined complex entities, preferably of propositional or first-order rather than higher-order level. These demands, however, encounter the following deep-rooted difficulty:

The new notions represent the horizontal (cut-wise) view of a fuzzy set (construed as a system of cuts), while usual fuzzy set theory represents fuzzy sets vertically (by membership degrees of its elements). Predicates in first-order fuzzy logic only formalize the vertical view of fuzzy sets; and although the latter can also be represented by systems of cuts, all usual FLd systems of first-order fuzzy logic require that the cuts be *nested.* This requirement is already built in the propositional core of common formal fuzzy logics, all of which presuppose the following principle (further on, we shall call it the principle *of persistence)*:

Principle of persistence: *If a proposition ϕ is guaranteed to be (at least) α-true, and then it is also guaranteed to be (at least) β-true for all $\beta \leq \alpha$.*

The principle is manifested, i.e., in the transitivity of implication, which is satisfied in all systems of FLd and is indispensable for multi-step logical deduction (more on this in Case Study 3 below). Since Dubois and Prade's gradual sets do not meet this requirement (the α-cuts need not be nested), the known systems of first-order fuzzy logic cannot represent them as fuzzy predicates. (Similarly, known systems of propositional fuzzy logic cannot represent them as fuzzy propositions.)

The reason why Dubois and Prade's notions depart so radically from the presuppositions of FLd resides in the conceptual difference between the approaches to fuzziness in FLd and FLt. In FLt, there are many possible interpretations of the meaning of mem- bership degrees. In particular (as stressed by Dubois and Prade in), fuzzy sets may in FLt represent imprecision *and* (Bˇehounek, 2005) membership degrees the gradual *change.* In FLd, however, membership degrees are only interpreted as guaranteed degrees of *truth;* and fuzzy sets in FLd represent the degree of satisfaction of truth conditions rather than interval-like imprecision. Thus in FLt, membership degrees can be understood as mere indices which parameterize the membership in a fuzzy set and which allow the gradual change from 0 to 1 ("fuzziness by fibering"). In FLd, truth degrees are what is preserved in graded inference, i.e., preserved w.r.t. The ordering *of* truth values; and this enforces the principle of persistence.

It should be noticed that the principle of cuts, which motivates the distinction between gradual elements and fuzzy sets in, is not itself alien to FLd. On the contrary—when following a certain FLd-sound methodology, many fuzzy counterparts of crisp notions do.

In re-interpreting the formulae of classical crisp definitions in many valued logic. If fuzzy notions are defined in this natural way, then the principle of cuts is often observed: the α- cuts of fuzzy sets are crisp sets, the α-cuts of fuzzy relations are crisp relations, the α-cuts of fuzzy Dedekind or MacNeille cuts come out as crisp Dedekind–MacNeille cuts, etc. Unlike in FLt, however, in FLd the fuzzy notions have also to confirm to the principle of persistence; this constrains the α-cuts to *nested* systems of the corresponding crisp objects. In the particular case of fuzzy elements, the α-cuts of an FLd fuzzy element a must not only be crisp elements (as in FLt), but also must satisfy the principle of persistence for all formulae, in particular for the formula $x = a$. The latter already necessitates that the α-cut of a equals its β-cuts for all $\beta \leq \alpha$; and since this should hold for all α, the fuzzy element a has to be constant. Thus in FLd we can only have constant fuzzy elements, which can be identified with ordinary crisp elements. Similarly, by enforcing the nesting of α-cuts, the principle of persistence reduces in FLd gradual sets to common fuzzy sets.

No doubt fuzzy elements are a natural notion, abundant in many real-life situations; therefore the above difficulties should not stop us from investigating them. There are no obstacles to investigating them in the framework of FLt. However, current FLd can only render them indirectly in a higher-order setting, since they do not conform to the principle of persistence upon which all current systems of FLd are founded. Thus even though (advanced) FLd can (clumsily) capture the new notions, they actually do not fall into its primary area; and so the way in which FLd can contribute to the investigation of these notion is rather limited.4 This of course does not diminish the importance of the new notions for FLt and does not even exclude the usefulness of their formal counterparts in some parts of FLd (Dubois, 2005).

The above analysis only shows that when employing fuzzy elements in FLd, we shall have to deal with complex objects (crisp functions from the set of internalized truth values) rather than some kind of more primitive notion.

A further analysis will be needed to find out if Dubois and Prade's gradual elements and sets can be treated propositionally or as a primitive first-order notion in a *radically new* system of deductive fuzzy

logic. Since a direct logical rendering of gradual sets would need to drop the principle of persistence, it would have to adopt an entirely different concept of truth preservation under inference; such a radical change would consequently affect virtually all logical notions.

Unfortunately, many straightforward approaches are not viable, as they would trivialize the theory. E.g., a notion of truth preservation based on the identity (rather than order) of truth degrees would reduce truth degrees to mere indices exactly in the way FLt does; however, it would trivialize the logic to classical Boolean logic.5 From the opposite point of view, this could be an indication that by treating membership degrees as mere indices (rather than truth degrees that should be preserved under graded inference), FLt does not in fact step out of the classical framework; it is the gradual inference what makes things genuinely fuzzy from the FLd point of view, rather than just employing some set of indices like [0, 1].

3. Case study 2: The entropy of a fuzzy set

Various definitions of the *entropy of* a fuzzy set have been proposed in traditional fuzzy mathematics, for instance:
- De Luca and Termini's entropy $Ek(A) = Dk(A) + Dk(Ac)$
- Yager's entropy $Yp(A) = 1 - \grave{p}(A, Ac)/(\grave{p}(A, \emptyset))p$
- Kaufmann's entropy $Kp(A) = 2n{-}1/p \cdot \grave{p}(A, A)$
- Kosko's entropy $Rp(A) = \grave{p}(A, A)/\grave{p}(A, \underline{A})$

where A is a finite [0, 1]-valued fuzzy set; Ac is its additive complement, $Ac(x) = 1 - A(x)$; A is defined as $A(x) = 1$ if $A(x) \geq 0.5$, and 0 otherwise; $\underline{A} = (A)c$; p, k are parameters, $p \geq 1$ and $k > 0$; $Dk(A) = -k\,Pi\,A(xi) \log A(xi)$; and $\grave{p}$ is the distance between finite fuzzy sets defined as $\grave{p}(A, B) = (Pi\,|A(xi) - B(xi)|)$.

The common feature of all such entropy measures is that they assign the minimal (zero) entropy to crisp sets, and maximal (unit) entropy to fuzzy sets with $A(x) = 0.5$ for all x in the universe of discourse.

The definition is motivated (and the name *entropy* justified) by the idea that the membership degree 0.5 tells us the least amount of information ("nothing") about the membership of x in A. In other words, that the membership degree of 0.5 gives us the same degree of "certainty" that x belongs to A as that x does not belong to A, and so it provides us with no information (knowledge) as to *whether x* belongs to A. The membership degrees of 0 and 1, on the other hand, give us full "knowledge" or "certainty" about the member- ship of x in A, and thus provide us with maximal information as regards the membership of x in A. The degree of fuzziness, measured by the entropy measures, thus (in FLt) expresses the informational contents contained in the fuzziness of the fuzzy set.

In FLd, on the other hand, such concepts of entropy do not have good motivation. This is because in FLd, the membership degree cannot be interpreted as the degree of *knowledge* or *certainty* of whether x belongs to A or not, but only as the degree of the (guaranteed) *truth* of the statement that x belongs to A. From the FLd point of view it is not true that $A(x) = 0.5$ gives us the least information on the membership in A. On the contrary each membership degree gives us the same (namely, full) information about the *extent of* membership in A (Cintula, 2006).

The difference between the information conveyed by membership degrees in FLt and FLd can be illustrated by the following consideration. We have the following trivial observation in all usual systems of FLd that contain a well-behaved implication connective $\Rightarrow$.

Fact 5, If it is provable that $A(z) \Rightarrow \phi(z)$ for all z, then for any membership degree α, if the truth degree of $A(x)$ is α, then the truth degree of $\phi(x)$ is at least α.

Thus in FLd, if we know that $x \in A$ is true to degree 0.5 and that all elements of A satisfy some property ϕ (in the sense of FLd—i.e., that $A(z) \Rightarrow \phi(z)$ is valid for all z), then we know that x satisfies ϕ at least to degree 0.5. Therefore in FLd, the truth degree of 0.5 does not represent "no knowledge" or "equal possibility of both cases". Rather, like any other membership degree, it represents a certain guaranteed degree of participation of x on the properties of A. In other words, any membership degree α of $x \in A$ tells us in FLd that the properties entailed by the membership in A will be satisfied by x at least to degree α.

From the informational point of view, in FLd (as shown by Fact 5) the membership degree 0.5 restricts the possible truth values of $\phi(x)$, for any property ϕ entailed by the membership in A, to the interval [0.5, 1]. In this sense, the least informative membership degree should in FLd be 0 (as it does not restrict the interval at all) and the most informative degree should be 1 (as it maximally restricts the interval to the single value 1). However, 0 is also the most informative (and 1 the least informative) degree as regards the satisfaction by x of the properties of another set, namely Ac. Therefore in FLd, the informational contents of membership degrees are not determined simply by their value.

Thus from the point of view of FLd, no membership degree conveys more information than another just by its value. Therefore, no concept of entropy which assigns the least informational contents to the fuzzy set with $A(x) = 0.5$ for all x is well-motivated in FLd. Consequently we have to conclude that the notions of entropy belong to the area of FLt rather than FLd; and even though they can be defined in higher-order FLd,8 their significance in FLd and the extent to which FLd can help investigate them is limited. This does not deny their importance and good motivation in FLt under the interpretations of membership degrees as indicated above (of knowledge, certainty, etc.); only they are not meaningful for the concept of

guaranteed truth, which is the domain of FLd. As stressed above, the unmotivatedness of the concept of entropy in FLd is caused by the fact that membership degrees represent in FLd the degrees of truth (of the statement "x ∈ A") rather than the degrees of knowledge or certainty about x ∈ A. The uncertainty about x ∈ A would not in FLd is expressed by an intermediate membership degree, but rather by an uncertain membership degree. The first idea how to render uncertain mem- bership degrees in FLd are, obviously, to take a crisp or fuzzy set of possible membership degrees, like in interval-valued fuzzy sets or type-2 fuzzy sets. However, in the framework of FLd, this idea has to be refined: a fuzzy set of membership degrees does not in FLd represent the degree of certainty or knowledge about the membership degrees, ei- ther, but only expresses the degree of truth of some property of membership degrees. Thus it would be necessary to introduce some modality, e.g., "it is known that", and interpret the fuzzy set of membership degrees α as expressing the truth degree of the statement "it is known that the membership degree of x ∈ A is α", rather than the degree of knowledge itself. This subtle difference is insignificant for atomic epistemic statements, but plays a role when considering complex epistemic statements composed by means of propositional connectives (for more on this distinction. The rendering of the uncertainty of membership in a fuzzy set, which motivates the notion of entropy in FLt, is thus in FLd much more complicated than what is expressed by simple intermediary membership degrees.

4. Case study 3: Aggregation of fuzzy data

The exclusive interpretation of membership degrees as guaranteed degrees of truth leads to certain restrictions on admissible logical systems of FLd. Since the intended interpretation "truth at least to α" is based on an order ("at least") of truth degrees, logical systems suitable for FLd have to be among the logics of partially ordered (or at least preordered) algebras or logical matrices, i.e., among Rasiowa's implicative logics or Cintula's weakly implicative logics. The property of prelinearity, advocated in as the characteristic feature of deductive *fuzzy* logics, then leads to Cintula's class of weakly implicative fuzzy logics. Another condition that further (de, 1972) constrains the class of logical systems best suitable for *deductive fuzzy* logic is the law of residuation. As will be shown in this section, the law of residuation and related requirements present another important difference between FLd and FLt.
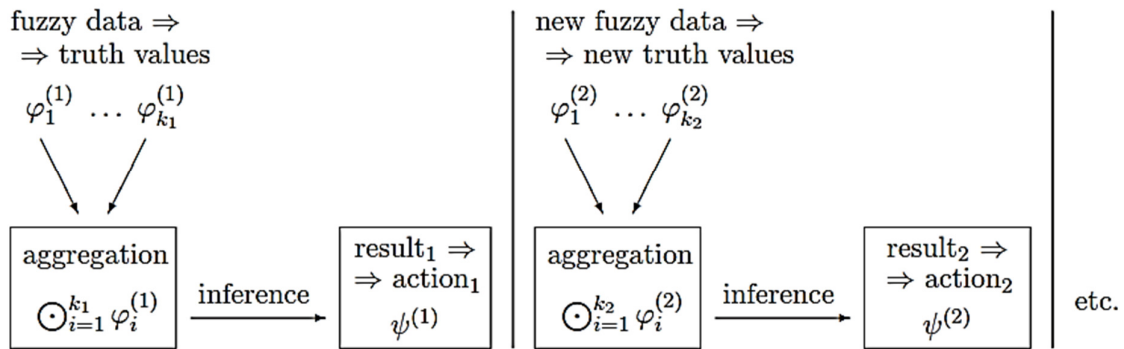

Fig. 1: Modus operandi of applied FLt

Another observation is that the data that enter the aggregation and inference are usually extra-logical (measured in the real world, read from a database etc.). In particular, they usually do not contain the operators J and → of the inference mechanism, and so in FLt inference one usually need not consider nested implications (the formulae expressing the inference laws are "flat"). The situation in FLd is different, as is the typical modus operandi of FLd.

The formally-deductive aims of FLd require the preservation of guaranteed truth values also in successive (iterated) inferences, which are typical for multiple-step deductions. In formal derivations we often have intermediary steps and results, lemmata, partial conclusions, etc., and we want the guaranteed truth degree of a conclusion to remain coherent through- out long deductions. Therefore, a typical modus operandi of FLd is the one depicted in Fig. 2.

Observe first that in the multiple-step derivations of FLd, the premises of the first steps still play a role in the following steps, since partial results enter further deductions. Furthermore, in the formally logical setting of FLd, formulae entering deductions need implication thus plays a double role in FLd deduction: it is used for making inferences, but can also occur as a connective within a formula that enters the inference as a premise or comes out as a conclusion. Similarly conjunction is used for the aggregation of premises, but can also appear as a connective inside the premises and conclusions. If both roles of the operators are to match, they have to satisfy conditions that describe the match of the roles. Namely, whenever $\phi1$ is a premise of implication (inference) and $\phi2 \rightarrow \psi$ is its

conclusion, both roles of implication will accord if $\phi1$ and $\phi2$ together (i.e., aggregated) imply ψ (since both $\phi1$ and $\phi2$ are after all premises for ψ—one in implication-as-inference and one in implication-as-connective); and vice versa, if $\phi1$ and $\phi2$ jointly imply ψ, then $\phi1$ alone should imply $\phi2 \rightarrow \psi$ (for the same reason). Similarly, $\phi1$ and $\phi2$ aggregated should imply ψ if and only if $\phi1 \; ^- \phi2$ implies ψ (this corresponds to the match of both roles for conjunction). Since by the earlier considerations implication-as-inference is in FLd understood as truth-preservation (i.e., the relation $\leq$), the requirement can be formulated as the condition

$$\phi1 \; ^- \phi2 \leq \psi \quad \text{if} \quad \phi1 \leq \phi2 \rightarrow \psi \qquad (1)$$

The general form for an arbitrary number of premises as in Fig. 2 already follows from (1). This law of *residuation is* therefore required in FLd for ensuring the coherence of the guaranteed truth thresholds in multiple-step deductions with nested implications, while it need not be required in one-step inferences with flat formulae in FLt. The principle of residuation restricts significantly the class of conjunctive operator's admissible in FLd.

Together with a few reasonable additional requirements (see Remark 10 below) it confines the FLd-suitable [0, 1]-conjunctions & to *left-continuous t-norms* (or residuated uninorms, if we admit degrees of full truth). Other operators for fuzzy data aggregation are not meaningful in FLd, though they are both meaningful and important in FLt (as FLt need not preserve guaranteed truth degree in nested and iterated inferences).

Like in the case of fuzzy elements and the notion of entropy, many FLt conjunctive operators are still definable in systems of deductive fuzzy logic: e.g., a broad class of t- norms which are *not* left-continuous is representable in the logic LÃ Π. Nevertheless, as in the cases above, the apparatus of FLd is most efficient for conjunctions to which the FLd systems are tailored, i.e., which respect the above constraints.

Remark 8, the constraints on admissible conjunction connectives rule out the meaningfulness of most cut-wise definitions in FLd. Since most left-continuous t-norms are not idempotent, α-cuts are in general not preserved by conjunction in most systems of FLd (except in G̈odel fuzzy logic of the minimum t-norm).11 Thus, e.g., the cut-wise definition of the intersection of fuzzy sets is from the FLd point of view only meaningful in G̈odel logic; in other systems of FLd, the cut-wise intersection (which equals the minimum- intersection) does not satisfy the defining condition of intersection that the membership degree of x in $A \cap B$ be the conjunction of the membership degrees of x in A and B, i.e., that $(A \cap B)x = Ax \;\&\; Bx$.

Thus, e.g., Dubois and Prade's definitions of elementary operations on gradual sets proposed in (which are cut-wise, as we noted in the first case study), can only be well- motivated in FLt. Similar considerations restrict the FLd-meaningfulness of many parts of categorical (sheaf) approach to fuzzy sets, which often works cut-wise (i.e., fiber-wise) and thus belongs to FLt rather than FLd.

Again, this does not diminish the importance of cut-wise notions in FLt; only we should be aware that they are not well-motivated in FLd. In deductive fuzzy logic, many cut-wise notions can still be defined, and some of them do have some importance even in FLd. For instance, in all logics based on continuous t-norms, the minimum conjunction $\wedge$ and maximum disjunction $\vee$ are definable, and by means of these connectives one can define the cut-wise operations of min-intersection and max-union. However, their role in FLd systems is different than that of the notions based on usual (strong) conjunction &; in particular, min-conjunction cannot be used as a surrogate for strong conjunction, since both connectives have different meaning. Strong conjunction & represents the use of *both* conjuncts, while min-conjunction $\wedge$ represents the use of any *one* of the conjuncts, as can be seen from the following equivalences valid in BL and related systems:

$$[(\phi1 \;\&\; \phi2) \rightarrow \psi] \quad \leftrightarrow \quad [\phi1 \rightarrow (\phi2 \rightarrow \psi)] \qquad (2)$$

$$[(\phi1 \;\wedge\; \phi2) \rightarrow \psi] \quad \leftrightarrow \quad [(\phi1 \rightarrow \psi) \;\vee\; (\phi2 \rightarrow \psi)] \qquad (3)$$

Since it is (2) that we need in iterated inference rather than (3), minimum conjunction cannot be used for aggregation of premises in FLd. Similarly, min-intersection does not represent membership in *both* fuzzy sets, but only in *any* of them, and cannot be used in contexts when both Ax *and* Bx are required. The following example demonstrates the methodological consequences of the distinction between the two conjunctions in FLd.

Example 9, the notion of anti-symmetry of a fuzzy relation R *w.r.t.* A similarity E defined with min-conjunction, i.e., by inf xy ($Rxy \wedge Ryx \rightarrow Exy$) as similarly in, is not well-motivated in FLd, since in antisymmetry we clearly need *both* Rxy and Ryx to infer *Exy* (neither *Rxy* nor *Ryx* alone is sufficient for

Exy in antisymmetric relations; cf. (3)). Thus in FLd, we have to define antisymmetry with strong conjunction, i.e., as inf xy ($Rxy \;\&\; Ryx \rightarrow Exy$), even though some theorems of will then fail.

From the *deductive point* of view, min-conjunction anti-symmetry is only well-motivated in model logic.

Remark 10, As mentioned at the beginning of this section, the requirements on the transmission of truth in FLd lead to the defining conditions of Rasiowa's implicative logics or Cintula's weakly implicative (fuzzy) logics. However, these conditions only ensure good behavior of *fully true* implication, which then corresponds to the order of truth degrees. Inferentially sound behavior of

partially true implication and conjunction requires further axioms, including the law of residuation (as seen above), since only then implication internalizes the transmission of partial truth and conjunction internalizes the cumulation of premises in graded inference. The latter law also makes formal systems of FLd belong to the well-known and widely studied class of substructural logics (in Ono's sense, i.e., the logics of residuated lattices).

Thus from the point of view of *deductive fuzzy logic*, Cintula's class of weakly im- plicative fuzzy logic is still too broad. Best suitable logics for FLd are only those weakly implicative fuzzy logics that satisfy residuation and several natural requirements of the internalization of local consequence (namely the logical axioms expressing the antitony resp. isotony of implication in the first resp. second argument, and associativity and com- mutativity of conjunction; cf.). The resulting class can be understood as the formal mathematical delimitation of deductive fuzzy logics. The above conditions characterize them as those weakly implicative fuzzy logics which are extensions or expansions of the logic UL of residuated uninorms or, if we add the law of weakening $\phi \to (\psi \to \phi)$, of the logic MTL of left-continuous t-norms.

Again this does not imply that other weakly implicative fuzzy logics or logics used in FLt are deficient. However, only logics from the above defined class suit best to the motivation of FLd (i.e., transmission of guaranteed partial truth in multi-step deductions) and admit the construction of formal fuzzy mathematics in the sense. This is because their implication and conjunction respectively internalize the local consequence relation and the cumulation of premises: they have, in Ono's words, a "deductive face". Other logics lie outside the primary area of interest of FLd, though they may be of their own importance and interest in FLt.
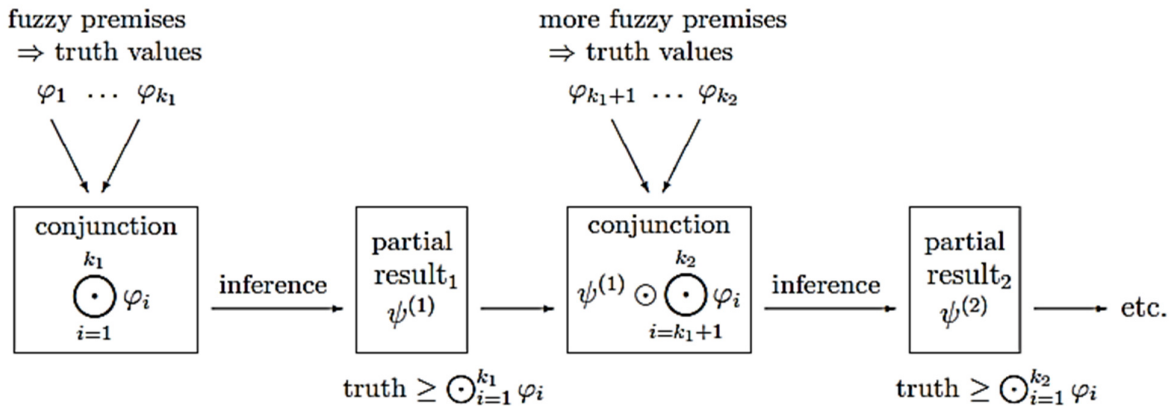

Fig. 2: Modus operandi of FLd

5. Conclusions

The three case studies show that FLd differs from broader FLt in many aspects, including the area of competence, methods, motivation, formalism, etc. It should perhaps be admit- ted those symbolic fuzzy logicians on the one hand and researchers in "mainstream fuzzy logic" on the other hand do rather different things and work in two distinct, even though related, areas (with some non-empty intersection). Since after the years of usage there is no chance for changing the name "fuzzy logic" in either tradition, a suitable determinative adjective (like *symbolic, formal, mathematical,* or as proposed here, *deductive*) attached to the name of the narrower and younger of both areas is probably the best solution to possible terminological confusions.

It is sometimes complained that fuzzy logic does not have a clear methodology for defining its notions and the direction of research. FLd, as its very narrow and specific branch, however, does possess a rather clear methodology, inherited for a large part from the methodology of non-classical logics and classical foundations of mathematics. This may be a consequence of the fact that FLd has chosen and clarified *one* of all possible interpretations of membership degrees, and now studies the properties of this single clarified concept. FLt, on the other hand, admits many interpretations of membership degrees and often tries to investigate them together, without separating them properly and without clarifying carefully which of the possible interpretations is considered.

A historical parallel can be seen in the early history of classical (crisp) set theory. As noticed by Kreisel, Cantor's notion of set was a mixture of at least three concepts finite sets of individuals, subsets of some domain, and properties (unbound classes). Part of the opposition against set theory was due to its confusion of these notions of set: the *crude mixture (*as Kreisel calls it) did not possess good properties, and the paradoxes of naive set theory confirmed the bad feeling. Only after one element of the crude mixture (viz iterated subsets) was clearly separated by Russell and Zermelo and shown to havegood and rich enough properties, the notion of set could start playing its foundational role in mathematics.

Similarly the theory of fuzzy sets presents a mixture of various *different* notions of fuzzy set

(truth-based, possibility, linguistic, frequentistic, probabilistic, etc.). While FLd has distilled one element of the mixture (namely the truth-based notion of fuzzy set), FLt often continues to investigate the crude mixture as a whole, only partially aware of the distinctions needed to be made. (Not that it never distinguishes the areas of applicability of its own notions: sometimes it does; but often it forgets to do so or is not careful enough.)

The methodological success of FLd and its advances should stimulate FLt to distin- guish with similar clarity the *exact* components of the crude mixture of notions of fuzzy set. Theoretical gains from their clear separation and investigation of the most promising ones would certainly be large (as was, e.g., the gain from conceptual and methodological clarification of the notion of probability); some areas of FLt besides FLd (e.g., possibility theory) already seem to be close to such clarification

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